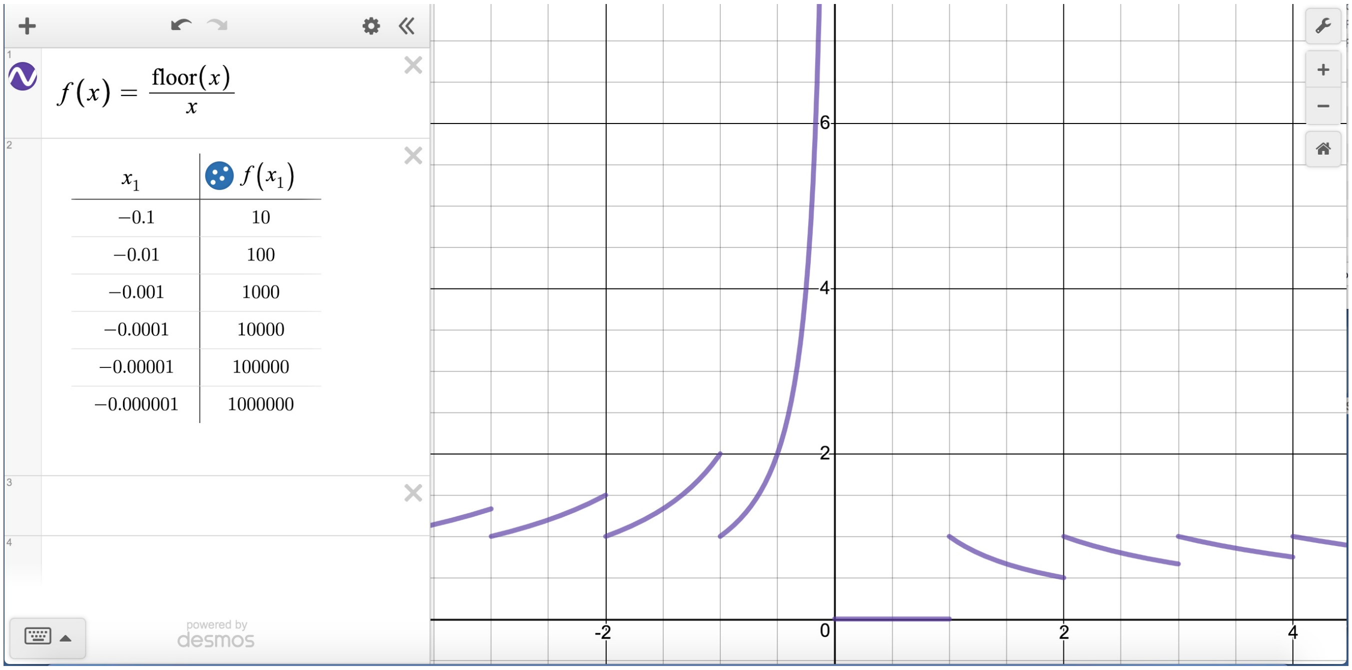
SECTION 2.4: INFINITE LIMITS

EXAMPLE 1: Let's take a closer look at $\displaystyle\lim_{x\to 0^-} \frac{\lfloor x \rfloor}{x}$.



We see as $x \to 0^-$, the values $f(x)$ become larger and larger, without bound. To see what's happening from a 'number sense' perspective, note that for $-1 < x < 0$, $\lfloor x \rfloor = -1$ so that as $x \to 0^-$,

$$\frac{\lfloor x \rfloor}{x} = \frac{-1}{x} \approx \frac{-1}{\text{'small' }(-)} = \text{'big' }(+)$$

The closer x gets to 0, the larger the outputs get. In this case, we write:

$$\lim_{x\to 0^-} \frac{\lfloor x \rfloor}{x} = \infty.$$

NOTE: Even though we write '$\displaystyle\lim_{x\to 0^-} \frac{\lfloor x \rfloor}{x} = \infty$,' please note that $\displaystyle\lim_{x\to 0^-} \frac{\lfloor x \rfloor}{x}$ **does not exist**.

By writing '$\displaystyle\lim_{x\to 0^-} \frac{\lfloor x \rfloor}{x} = \infty$', we are just being specific about **how** the limit **fails** to exist.

IN GENERAL:

- We write $\displaystyle\lim_{x\to a^-} f(x) = \infty$ to mean that as $x \to a^-$, the values $f(x)$ increase without bound.
- We write $\displaystyle\lim_{x\to a^+} f(x) = \infty$ to mean that as $x \to a^+$, the values $f(x)$ increase without bound.
- We write $\displaystyle\lim_{x\to a} f(x) = \infty$ if both $\displaystyle\lim_{x\to a^-} f(x) = \infty$ **and** $\displaystyle\lim_{x\to a^+} f(x) = \infty$.

NOTE: If function values **decrease** without bound in any of the situations above, we use the symbol '$-\infty$.'

VERTICAL ASYMPTOTES: If one of $\displaystyle\lim_{x\to a^-} f(x) = \infty$, $\displaystyle\lim_{x\to a^+} f(x) = \infty$, or $\displaystyle\lim_{x\to a} f(x) = \infty$, the line $x = a$ is called a **vertical asymptote** to the graph of $y = f(x)$.

EXAMPLE 2: Use limits to find the vertical asymptotes and holes in the graph of $f(x) = \dfrac{x^2 - 2x - 3}{x^2 - 1}$.

Since vertical asymptotes come from infinite limits, and the latter come from gaps in the domain, we first find where f is undefined. In this case, we set the denominator to zero: $x^2 - 1 = 0$ or $x = \pm 1$.

Next, we analyze f near these excluded values using limits.

NEAR $x = -1$**:** Substituting $x = -1$ into $f(x)$ gives a '$\frac{0}{0}$' indeterminate form so we try factoring and cancelling:

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 - 2x - 3}{x^2 - 1} = \lim_{x \to -1} \frac{\cancel{(x+1)}(x-3)}{\cancel{(x+1)}(x-1)} = \frac{-1-3}{-1-1} = 2$$

(Which agrees with our numerical and graphical guess from Section 2.2!)

Hence, the graph of $y = f(x)$ has a **hole** at $(-1, 2)$.

NEAR $x = 1$**:** Substituting $x = 1$ into $f(x)$ gives '$\frac{-4}{0}$' so we know there is unbounded behavior happening.

As $x \to 1^-$, $f(x) \approx \dfrac{-4}{\text{'small' } (-)} = \text{'big' } (+)$, so $\displaystyle\lim_{x \to 1^-} f(x) = \infty$.

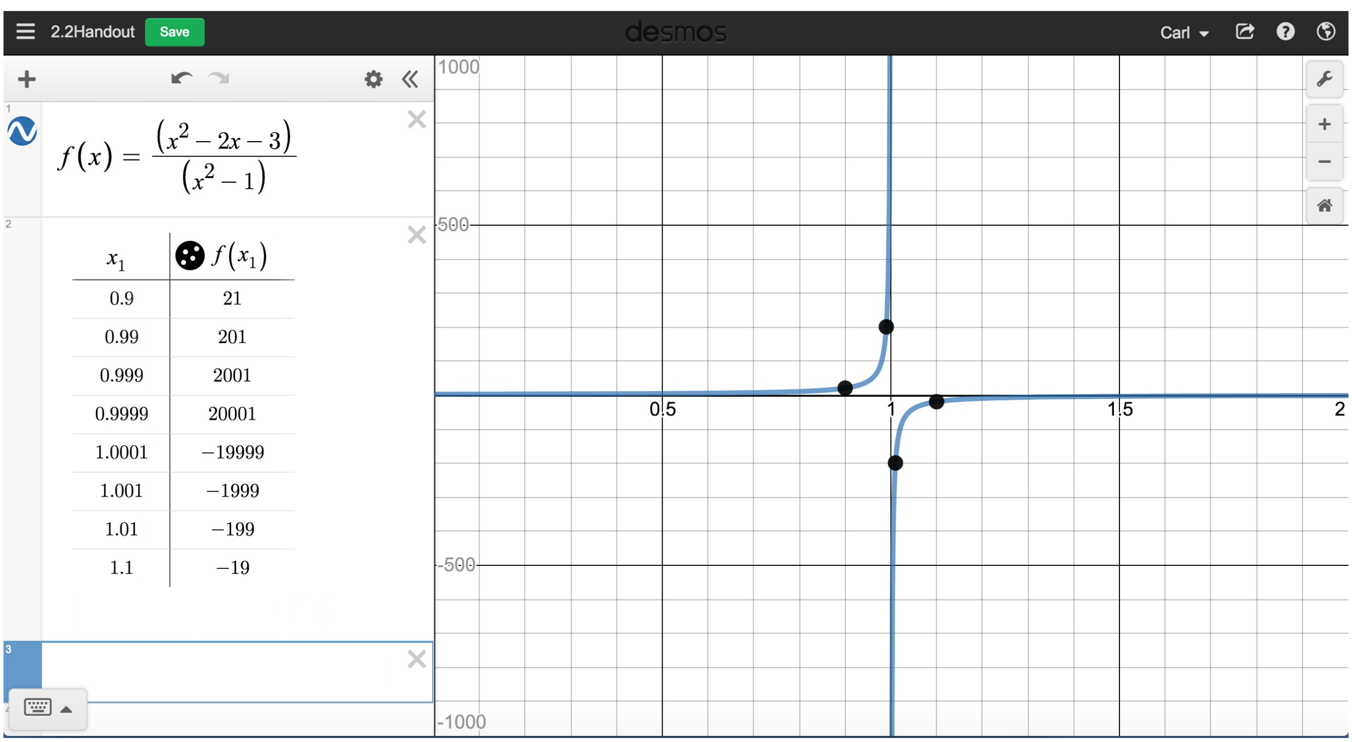
Hence, $x = 1$ is a vertical asymptote to the graph.

As $x \to 1^+$, $f(x) \approx \dfrac{-4}{\text{'small' } (+)} = \text{'big' } (-)$, so $\displaystyle\lim_{x \to 1^-} f(x) = -\infty$.

This means the graph is also asymptotic to $x = 1$ from the right.

NOTE: Since we have 'mismatched' unbounded behavior near $x = 1$, we write '$\displaystyle\lim_{x \to 1} f(x)$ does not exist.'

We confirm our results graphically using desmos.



EXAMPLE 3 (VIDEO): Use limits to find the vertical asymptotes and holes in the graph of the given function.

1. $f(x) = \dfrac{1-3x}{(x-2)^2}$ — Ans: VA: $x = 2$.

2. $f(x) = x^{-2/3}$ — Ans: VA: $x = 0$.

3. $f(t) = \cot(t)$ — Ans: VAs: $t = k\pi$ where $k = 0, \pm 1, \pm 2, \ldots$

4. $f(t) = \dfrac{\sin(2t)}{t^2 - 2t}$ — Ans: Hole: $(0, -1)$; VA: $t = 2$

5. $f(x) = \dfrac{2x}{\sqrt{4x - x^2}}$ — Ans: Hole: $(0, 0)$; VA: $x = 4$

EXAMPLE 4 (VIDEO): Sketch the graph of a function which satisfies all of the following criteria:

- $\lim\limits_{x \to -2^-} f(x) = \infty$
- $\lim\limits_{x \to -2^+} f(x) = 1$
- $f(-2) = 3$
- $\lim\limits_{x \to 0} f(x) = -\infty$

HOMEWORK: Section 2.4: 1 - 53 odd, 55*, 67*